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MODIFIED NEW EXTENDED WEIBULL DISTRIBUTION

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ABSTRACT

This article is devoted to study a new four-parameter model, called Modified New Extended Weibull distribution, that exhibits either increasing, unimodal or modified unimodal shaped failure rates. Some of its statistical properties such as moments, quantile, generating functions and densities of the order statistics are obtained. The method of maximum likelihood will be used, for estimating the model parameters. The proposed model will be illustrated, by analyzing a real data set, and goodness of fit result of the proposed model will be compared with Weibull and four of its foremost modifications, including new extended Weibull (NEx-W), Flexible Weibull extension (FWEx), generalized power Weibull (GPW) and Kumaraswamy generalized power Weibull (Ku-GPW) distributions.

KEYWORDS: Flexible Weibull Extension, Unimodal and Modified Unimodal Failure Rates, Moment Generating Function, Order Statistics, Maximum Likelihood Estimates

1. INTRODUCTION

The Weibull model, introduced by Weibull [10], is a popular statistical model for modeling lifetime data, where the hazard rate function is monotonic. Recently appeared new classes of distributions were based on, modifications of the Weibull model, to provide a good fit to lifetime data sets with monotonic failure rate. These modifications, including exponentiated Weibull (EW), introduced by Mudholkar and Srivastava [9], the additive Weibull (AW) proposed by Xie et al. [11], the modified Weibull extension (MW Ex) introduced by Xie et al. [22], the new modified Weibull (NMW) introduced by Lai et al. [13], flexible Weibull extension (FWEx) of Bebbington et al. [8], Generalized Flexible Weibull Extension (GFWEx) proposed by Ahmad and Iqbal [6], and other recent extensions of the Weibull model, proposed by Ahmad and Hussain [1], [2] and [4]. Recently, Ahmad and Hussain [3] proposed a new extension of the Weibull model, called NEx-W has the cumulative distribution function (CDF), given by

$$G(z;\alpha,\beta,\sigma)=1-e^{-e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^{2}}\right)}},\quad z,\alpha,\beta,\sigma>0. \tag{1}$$

The density function corresponding to (1), given by

$$g(z;\alpha,\beta,\sigma)=\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^{3}}\right)e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^{2}}\right)}\,e^{-e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^{2}}\right)}}. \tag{2}$$

The survival function (SF) of the NEx-W distribution is

$$S(z;\alpha,\beta,\sigma)=e^{-e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^{2}}\right)}},$$

with hazard function (HF) given by

$$h(z;\alpha,\beta,\sigma)=\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}.$$

In this article, a new generalization of the Weibull model named as Modified New Extended Weibull (MNEx-W), is introduced by mixing up two components. Where, one component has the NEx-W model, while the other has the exponential model. The CDF of the exponentially distributed random variable, is given by

$$G(z;\lambda)=1-e^{-\lambda z},\quad z,\lambda>0.$$

Based on this density, by replacing z with $-\log(1-G(z))$, extensions of the Weibull model are defined by (see Ahmad and Iqbal [6] and Ahmad and Hussain [2]). Where, $-\log(1-G(z))$ satisfies some conditions, see Ahmad and Hussain [5]. So,

$$F(z)=\int_0^{-\log(1-G(z))}\lambda e^{-\lambda z}\,dz,$$

$$F(z)=1-(1-G(z))^{\lambda}. \tag{3}$$

The density corresponding to (3), given by

$$F(z)=\lambda g(z)(1-G(z))^{\lambda-1}. \tag{4}$$

By using the CDF and PDF of NEx-W distribution in (3), and in (4), one may have the CDF and the density of the proposed distribution. The proposed distribution is very flexible and is able to model life time data with increasing, unimodal or modified unimodal failure rates. The rest of the article is structured in the following manner: Section 2, defines MNEx-W distribution. The basic statistical properties of the proposed model are discussed in Section 3. Section 4 and 5, covers the moment generating function and factorial moment generating function, of the proposed model. The densities of the order statistics are derived in section 6. The estimation of the model parameters is discussed in section 7. Section 8, offers analysis of a real data set. Finally, section 9, provides the concluding remarks.

2. MODIFIED NEW EXTENDED WEIBULL DISTRIBUTION

The CDF of the MNEx-W distribution is defined by the following expression

$$F(z;\alpha,\beta,\sigma,\lambda)=1-e^{-\lambda e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}},\quad z,\alpha,\beta,\sigma,\lambda>0. \tag{5}$$

The density function associating to (5), is given by

$$f(z;\alpha,\beta,\sigma,\lambda)=\lambda\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}\,e^{-\lambda e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}}.$$

The SF of the MNEx-W distribution is

$$S(z;\alpha,\beta,\sigma,\lambda)=e^{-\lambda e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}},$$

With HF,

$$h(z;\alpha,\beta,\sigma,\lambda)=\lambda\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}. \quad \textbf{(6)}$$

The possible shapes of the HFs of MNEx-W model is provided, for four parameter combinations of the model in figure 1 & figure 2. The shapes in figure 1& 2, for σ =0.5 and different values of α,β and $\lambda,$ displays that, the HF of MNEx-W distribution can either be increasing unimodal, or modified unimodal shaped failure rates:

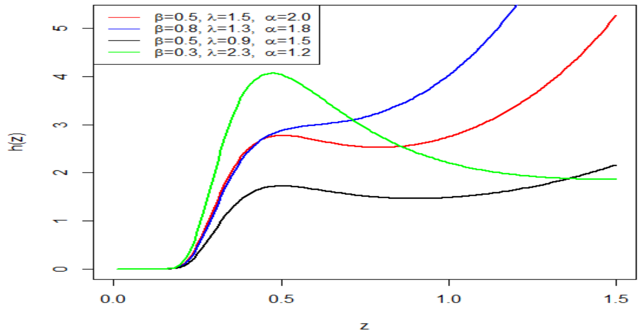

Figure 1: HF of the MNEx-W Distribution, for a Choice Values of Parameters

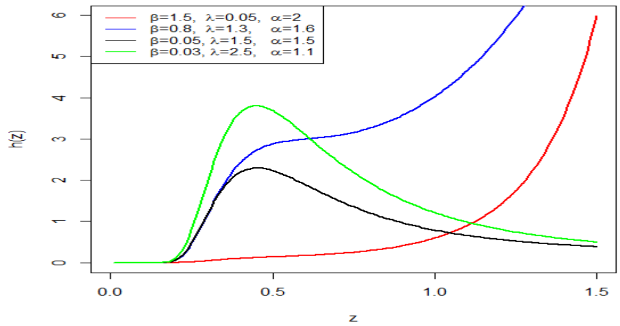

Figure 2: HF of the MNEx-W Distribution, for a Choice Values of Parameters

3. BASIC PROPERTIES

This section of the papersderives the basic statistical properties of the MNEx-W distribution.

3.1. Quantile and Median

The expression for the q^{th} quantile z_q of the MNEx-W distribution is given by

$$\beta z_q^{\alpha}-\frac{\sigma}{z_q^2}-\log\left\{-\frac{\log(1-q)}{\lambda}\right\}=0. \quad \textbf{(7)}$$

Using q=0.5, in (7), we have the median of the MNEx-Wdistribution. Also, by setting q=0.25, and q=0.75, in (7), we obtain the 1st and 3rd quartiles of the MNEx-W distribution, respectively.

3.2. Generation of Random Numbers

The formula for generating random numbers from MNEx-W distribution is given by

$$\beta z^{\alpha}-\frac{\theta}{z^{2}}-\log\left\{-\frac{\log(1-R)}{\lambda}\right\}=0, \qquad R\sim U(0,1).$$

3.3. Moments

If Z has the MNEx-W with parameters $(\alpha,\beta,\sigma,\lambda)$, then the r^{th} moments of Z is derived as

$$\mu_r^{/}=\int_0^{\infty} z^r g(z;\alpha,\beta,\sigma,\lambda)\,dz,$$

$$\mu_r^{/}=\lambda\int_0^{\infty} z^r\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}e^{-\lambda e^{\left(\beta z^{\gamma}-\frac{\sigma}{z^2}\right)}}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\frac{(-1)^i(\lambda)^{i+1}}{i!}\int_0^{\infty} z^r\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)\left\{e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}\right\}^{i+1}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\frac{(-1)^i(\lambda)^{i+1}}{i!}\int_0^{\infty} z^r\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)\left\{e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}\right\}^{i+1}dz,$$

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(i+1)^j\alpha\beta^{j+1}(\lambda)^{i+1}}{i!\,j!}\int_0^{\infty} z^{r+\alpha-2j-1}e^{\beta(i+1)z^{\alpha}}dz$$
$$+2\sigma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(i+1)^j\beta^j(\lambda)^{i+1}}{i!\,j!}\int_0^{\infty} z^{r-2(j+1)-1}e^{\beta(i+1)z^{\alpha}}dz,$$

Finally, the following expression is observed

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(i+1)^j\beta^{j+1}(\lambda)^{i+1}}{i!\,j!}\frac{\Gamma\left(\frac{r-2j}{\alpha}+1\right)}{(\beta(i+1))^{\frac{r-2j}{\alpha}+1}}$$
$$+2\sigma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^i(i+1)^j\beta^j(\lambda)^{i+1}}{i!\,j!}\frac{\Gamma\left(\frac{r-2(j+1)}{\alpha}\right)}{\alpha(\beta(i+1))^{\frac{r-2(j+1)}{\alpha}}}. \qquad \textbf{(8)}$$

4. MOMENT GENERATING FUNCTION

If Z~MNEx-W$(z;\alpha,\beta,\sigma,\lambda)$, then the formula for the moment generating function (MGF) of Z is derived as

$$M_z(t)=E\left(e^{tz}\right)$$

$$M_z(t)=\int_0^{\infty} e^{tz}g(z;\alpha,\beta,\sigma,\lambda)\,dz,$$

$$M_z(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{/}. \qquad \textbf{(9)}$$

Using (8), in (9), one can easily derive the MGF of MNEx-Wdistribution.

5. FACTORIAL MOMENT GENERATING FUNCTION

If Z~MNEx-W$\left(z;\alpha,\beta,\sigma,\lambda\right)$, then the factorial moment generating function (FMGF) of Z can be derived as

$$H_0\left(\delta\right)=E\left(\left(1+\delta\right)^z\right)$$

$$H_0\left(\delta\right)=E\left(e^{z\ln(1+\delta)}\right),$$

$$H_0\left(\delta\right)=\int_0^{\infty}e^{z\ln(1+\delta)}g\left(z;\alpha,\beta,\sigma\right)dz,$$

$$H_0\left(\delta\right)=\sum_{r=0}^{\infty}\frac{\left(\ln^r\left(1+\delta\right)\right)}{r!}\mu_r^{/}. \tag{10}$$

Using (8), in (10), we can derive the FMGF of MNEx-W distribution.

6. ORDER STATISTICS

Let $Z_{1,}\ Z_2\ ...Z_k$ are independently and identically distributed (i.i.d) random variables taken from MNEx-W $\left(z;\alpha,\beta,\sigma\right)$, in such a way that $Z_{(1:k)}\leq\ .\ .\ .\leq Z_{(k:k)}$. So, the density function of $Z\left(i:k\right)$, i=1, 2, 3, . . . , k is

$$g_{i:k}\left(z\right)=\frac{1}{Beta\left(i,k-i+1\right)}g\left(z;\Phi\right)\left[G\left(z;\Phi\right)\right]^{i-1}\left[1-G\left(z;\Phi\right)\right]^{k-i}. \tag{11}$$

Where, $\Phi=\left(\alpha,\beta,\sigma,\lambda\right)$

The density function of the minimum order statistics is derived as

$$g_{1:k}\left(z\right)=k\,g\left(z\right)\left[1-G\left(z\right)\right]^{k-1}.$$

$$g_{1:k}\left(z\right)=k\lambda\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}\left\{e^{-\lambda e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}}\right\}^k. \tag{12}$$

Also, the density of the maximum order statistics is derived as

$$g_{k:k}\left(z\right)=k\,g\left(z\right)\left[G\left(z\right)\right]^{k-1},$$

$$g_{k:k}\left(z\right)=k\lambda\left(\alpha\beta z^{\alpha-1}+\frac{2\sigma}{z^3}\right)e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}\ e^{-\lambda e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}}\left\{1-e^{-\lambda e^{\left(\beta z^{\alpha}-\frac{\sigma}{z^2}\right)}}\right\}^{k-1}. \tag{13}$$

7. ESTIMATION

This section determines the maximum likelihood estimates of the model parameters. Let $Z_1,Z_2,\ ...Z_k$ are drawn rparameters, thenx-W with parameters $(\alpha,\beta,\sigma,\lambda)$, then the log-likelihood function of this sample is

$$\ln L = k\log\lambda + \sum_{i=1}^{k}\log\left(\alpha\beta z_i^{\alpha-1} + \frac{2\sigma}{z_i^3}\right) + \sum_{i=1}^{k}\left(\beta z_i^{\alpha} - \frac{\sigma}{z_i^2}\right) - \lambda\sum_{i=1}^{k} e^{\left(\beta z_i^{\alpha} - \frac{\sigma}{z_i^2}\right)}. \quad \textbf{(14)}$$

Obtaining the partial derivatives of the expression in (14), on a parameter, and then equating the result equal to zero, one may derive the following results

$$\frac{d\ln L}{d\beta} = \sum_{i=1}^{k}\frac{\alpha z_i^{\alpha-1}}{\left(\alpha\beta z_i^{\alpha-1} + \frac{2\sigma}{z_i^3}\right)} + \sum_{i=1}^{k} z_i^{\alpha} - \lambda\sum_{i=1}^{k} z_i^{\alpha} e^{\left(\beta z_i^{\alpha} - \frac{\sigma}{z_i^2}\right)}. \quad \textbf{(15)}$$

$$\frac{d\ln L}{d\alpha} = \beta\sum_{i=1}^{k}\frac{\left(\alpha z_i^{\alpha-1}\log(z_i) + z_i^{\alpha-1}\right)}{\left(\alpha\beta z_i^{\alpha-1} + \frac{2\sigma}{z_i^3}\right)} + \beta\sum_{i=1}^{k} z_i^{\alpha}\log(z_i) - \lambda\beta\sum_{i=1}^{k} e^{\left(\beta z_i^{\alpha} - \frac{\sigma}{z_i^2}\right)} z_i^{\alpha}\log(z_i). \quad \textbf{(16)}$$

$$\frac{d\ln L}{d\sigma} = \sum_{i=1}^{k}\frac{\frac{2}{z_i^3}}{\left(\alpha\beta z_i^{\alpha-1} + \frac{2\sigma}{z_i^3}\right)} - \sum_{i=1}^{k}\frac{1}{z_i^2} + \lambda\beta\sum_{i=1}^{k}\frac{e^{\left(\beta z_i^{\alpha} - \frac{\sigma}{z_i^2}\right)}}{z_i^2}. \quad \textbf{(17)}$$

$$\frac{\ln L}{d\lambda} = \frac{k}{\lambda} - \sum_{i=1}^{k} e^{\left(\beta z_i^{\alpha} - \frac{\sigma}{z_i^2}\right)}. \quad \textbf{(18)}$$

From (15)-(19), it is quite clear that these expressions are not in closed forms, and cannot be solved manually. The statistical software can be utilized to solve them numerically by deploying the iterative techniques such as, the Newton-Raphson algorithm. The “SANN” algorithm is used in R language, to derive the numerical estimates of the model parameters.

8. APPLICATION

To illustrate the significant improvement of the proposed model, an example is presented. The goodness of the fit result of the suggested model is compared with that of five other well-known lifetime models. The investigative measures, including Akaike’s Information Criterion (AIC), corrected Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC) and log likelihood $-2l(.,z)$. On deciding upon these measures, it is proved that the proposed model provides greater distributional flexibility.

Example: 1

The data set taken Bader and Priest [7], containing the single fibers of 20 mm, with a sample of size 63. The data are: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395 and 5.020. The selected criteria of the MNEx-W, NEx-W, W, FWExD, GPW and Ku-GPW distributions are summarized in table 1.

Table 1: Goodness of Fit Results for MNEx-W, NEx-W, W, FWEx, GPW and Ku-GPW

Dist.	Max. Likelihood Estimates	$-2logl$	AIC	CAIC	HQIC
MNEx-W	$\hat{\alpha}$ =0194, $\hat{\beta}$ =1.071, $\hat{\sigma}$ =13.95, $\hat{\lambda}$ = 1.023	54.347	113.987	114.465	117.324
NEx-W	$\hat{\alpha}$ =0.254, $\hat{\beta}$ =1.711, $\hat{\sigma}$ =23.65	56.474	118.949	119.356	121.478
W	$\hat{\alpha}$ =5.049, $\hat{\beta}$ =3.315	61.957	127.914	128.114	129.445
FWEx	$\hat{\beta}$ = 0.307 , $\hat{\sigma}$ = 1.396	61.592	127.19	127.399	128.880
GPW	$\hat{\alpha}$ =3.151 , $\hat{\beta}$ =19.824 , $\hat{\sigma}$ =179.363	69.271	144.542	144.949	147.071
Ku-GPW	$\hat{a}$ =40.07, $\hat{b}$ =1.41, $\hat{\alpha}$ =0.67, $\hat{\beta}$ =2.21, $\hat{\sigma}$ =0.46	56.946	122.692	123.745	126.907

9. CONCLUSIONS

In this paper, a new model called modified new extended Weibull distribution, has been proposed and its properties are studied. The proposed model is introduced by mixing up two components, one with new extended Weibull model and the other with an exponential distribution. The idea is to add an additional parameter to new extended Weibull distribution, to introduce a more flexible model, so that the hazard function is either increasing, unimodal or modified unimodal shaped. Finally, by analyzing a real data set, it is shown that, the proposed distribution provides a better fit than Weibull and some of its well-known existing modifications.

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